Robust Mesh Motion Based on Monge-Kantorovich Equidistribution

Luis Chacón, ORNL; Gian Luca Delzanno, John M. Finn, T-5

A method of grid adaptation using Monge-Kantorovich optimization has recently been developed [1]. This method enforces equidistribution of an error estimate of a discretized partial differential equation (PDE), based on the principle that the total error is minimized if the grid is chosen to equidistribute the local error. In 1D this condition determines the grid uniquely; in higher dimensions, equidistribution alone does not determine the grid uniquely. In Monge-Kantorovich theory, one minimizes

$$W \equiv \int_X \rho(\mathbf{x})|\mathbf{x}' - \mathbf{x}|^2 d\mathbf{x}$$

enforcing equidistribution as a constraint. The variables $\mathbf{x}(\zeta)$ and $\mathbf{x}'(\zeta)$ are mapped from the logical or computational space Ξ, which is the unit square (unit cube in 3D) with a uniform grid, to the physical space X, which can be fairly arbitrary [2]. For time-stepping applications, the images $\mathbf{x}(\zeta)$ and $\mathbf{x}'(\zeta)$ for a uniform grid on Ξ represent the adapted grid on X at two different times. Equidistribution is enforced by requiring that the Jacobian $J \equiv \det(\partial x'_i/\partial x_j) = \rho(\mathbf{x})/\rho'(\mathbf{x}')$, where ρ and ρ' are error estimates at the two times. Minimizing W with the equidistribution constraint leads to the result

$$\mathbf{x}' = \nabla\Phi(\mathbf{x})$$

and substituting this gradient condition into the equidistribution condition leads to the classic *Monge-Ampère equation* [1] or MA equation

$$\nabla^2\phi + \phi_{xx}\phi_{yy} - \phi_{xy}^2 = \frac{\rho(\mathbf{x})}{\rho'(\mathbf{x}')} - 1$$

where $\Phi(\mathbf{x}) = \mathbf{x}^2/2 + \phi(\mathbf{x})$. Notice that nonlinearity enters in two ways: in the quadratic terms on the left (the Hessian), and in the denominator on the right (where $\mathbf{x}' = \mathbf{x} + \nabla\phi$). Neumann boundary conditions $\hat{\mathbf{n}} \cdot \nabla\phi = 0$ on the boundary ∂X are used, to assure that boundary points move to boundary points. The operator on the left with these boundary conditions has a null space $\phi \to \phi + C$. Correspondingly, we must require that the right-hand side be in its range, i.e., $\int_V [\rho(\mathbf{x})/\rho'(\mathbf{x}') - 1]\, dV = 0$. This is the nonlinear analog of the solvability condition $\int \rho dV = 0$ in solving the Poisson equation with Neumann boundary conditions. This nonlinear solvability condition, which depends on the map $\mathbf{x}'(\mathbf{x})$ at each Newton iteration, must be applied on each iteration for convergence. In [1] it was shown that, for monitor functions ρ and ρ' whose variation is moderate, solutions to the Monge-Ampère equation also minimize the mean distortion, defined as the integral over X of the trace of

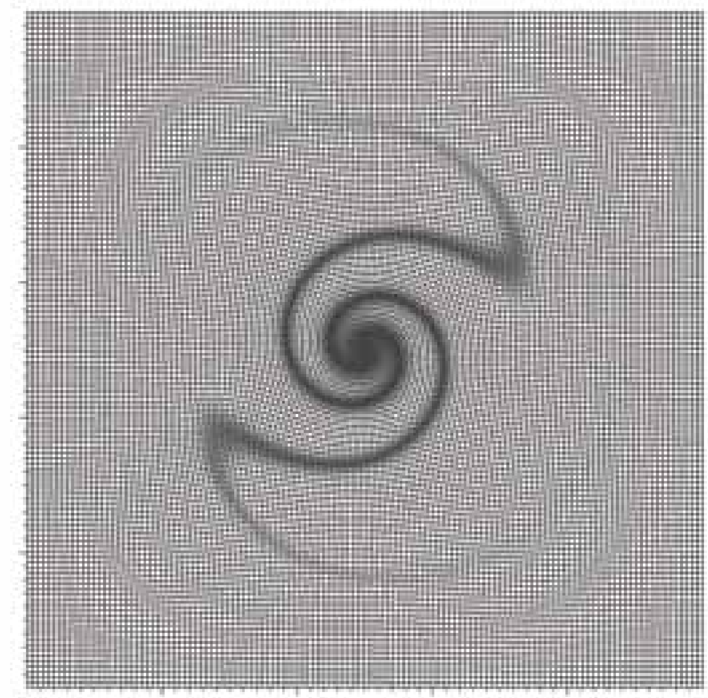

Fig. 1. Snapshot of the direct mesh at t = 90.

the covariant metric tensor. Grid folding occurs when the distortion grows to a degree that the sides of a cell intersect. In this sense, grids obtained by the Monge-Kantorovich method are robust to tangling.

For time-stepping applications, this approach to grid adaptation can be accomplished in two distinct manners. The first is the *sequential* approach, in which ρ and ρ' are error estimates at two successive time steps. In the second method, the *direct* method, ρ is the error estimate at the initial time and ρ' is again the error estimate at the next time step. The reason these two approaches give different grids is that a composition of gradient maps is not a gradient map. (The inverse map, on the other hand, is a gradient map $\mathbf{x} = \nabla_{\mathbf{x}'}\Psi$, where Φ and Ψ are Legendre transforms.) We have tested both the sequential method and the direct method for several examples in order to determine the quality of the grids produced, specifically the robustness to distortion that leads to folding or tangling.

The PDE used was the passive scalar equation for a given incompressible flow

$$\frac{\partial \chi}{\partial t} + \mathbf{v} \cdot \nabla \chi = 0$$

Rather than solve a discretized version of this passive scalar equation for χ using an error estimate, we have solved it for several relatively simple flows by the method of characteristics and proceeded to equidistribute χ *itself*. Equidistributing χ does not test the linking of the moving mesh approach with an error estimate, which is a goal for future work. Rather, equidistribution of χ was chosen as an efficient way to test the relation between the grids determined by the sequential and direct methods, respectively. That is, for the sequential method, we have $\rho(\mathbf{x}) = \chi(\mathbf{x},t)$ and $\rho'(\mathbf{x}) = \chi(\mathbf{x},t + \Delta t)$; the direct method takes $\rho(\mathbf{x}) = \chi(\mathbf{x},0)$ and $\rho'(\mathbf{x}) = \chi(\mathbf{x},t + \Delta t)$.

One of the example computations we have performed uses differential rotation $\Omega(r) = 16\Omega_0 \max[r(0.5 - r),0]$ on $X = (0,1) \times (0,1)$, with $\Omega_0 = -0.1$. The grid obtained with the direct method is shown in Fig. 1 at $t = 90$. Figure 2 shows the equidistribution error, the mesh distortion, and the maxima of the absolute values of the x- and y-components of the grid velocity. These quantities are shown for both the sequential method and for the direct method. Clearly, the direct method is better with respect to equidistribution and distortion, and also has somewhat smaller grid velocities. After $t \approx 90$, the sequential method begins to lose accuracy and soon breaks. The direct method continues well past this point, giving a smooth mesh even when the passive scalar structures are much smaller than can be resolved on the grid. Based on this example and several other examples, we conclude that the sequential method works fairly well for short times, but distortions in the grid eventually develop. The direct method appears to be completely robust, and in our view is the method of choice.

For more information contact John M. Finn at finn@lanl.gov.

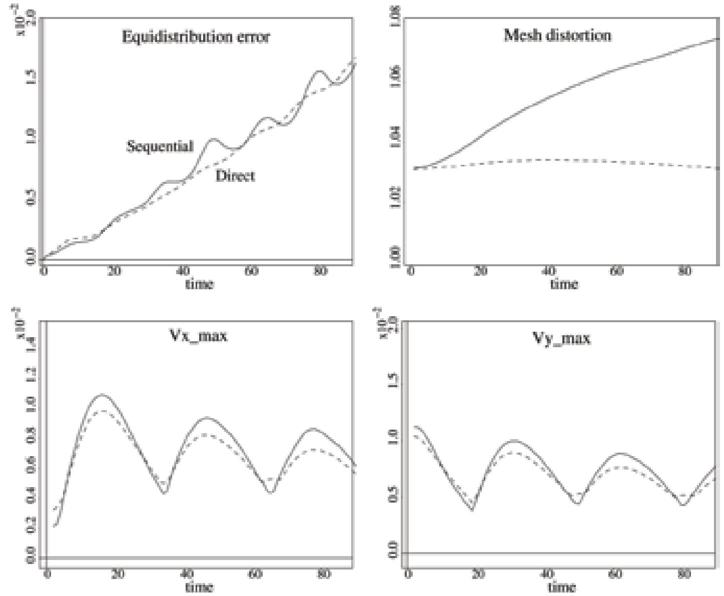

Fig. 2. Time histories of the equidistribution error, the mean mesh distortion, and the maxima of the absolute values of the grid velocity components.

[1] G.L. Delzanno, L. Chacon, J.M. Finn, Y. Chung and G. Lapenta, *J. Comp. Phys.* **227**, 9841 (2008).
[2] J.M. Finn, G.L. Delzanno, L. Chacón, *Proc. 17th Int. Meshing Roundtable,* Springer-Verlag (2008).

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